# October Math Gems

Problem of the week 22

## §1 Problems

**Problem 1.1.** If $x^2 = 2023 + y$ ,$y^2 = 2023 + x$ where $x, y \in R$, $x \neq y$
Then, find the value of $xy$

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**Problem 1.2.** Evaluate the following series

$$\sum_{n=1}^{1023} \log_2(1 + \frac{1}{n})$$

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**Problem 1.3.** If $\tan(x) - \tan^2(x) = 1$, Then, the value of the following expression is equal to

$$\tan^4(x) - 2\tan^3(x) - \tan^2(x) + 2\tan(x) + 1 =$$

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**Problem 1.4.** The measure of a regular polygon's interior angle is 4 times bigger than the measure of its internal angle. How many sides does the polygon have?

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**Problem 1.5.** How many sides does a convex polygon have if all its external angles are obtuse?

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**Problem 1.6.** show that the triangle $ABC$ where $\frac{a+c}{b} = cot\frac{B}{2}$ is right-angled

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**Problem 1.7.** Show that, if in triangle $ABC$ we have $\cot(A) + \cot(B) = 2\cot(C)$ ,then $a^2 + b^2 = 2c^2$

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**Problem 1.8.** Known that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

The minimum value of

$$3^{\sin^6(x)} + 3^{\cos^6(x)}$$

can be written in the form of $ab^c$. Find $4cb + a$

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**Problem 1.9.** show that in any right-angle triangle $ABC$ we have $\tan\frac{A-B}{2}\tan\frac{C}{2} = \frac{a-b}{a+b}$, where $A + B = C$

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**Problem 1.10.** If

$$\sum_{r=0}^{n-1} \log_2\left(\frac{r+2}{r+1}\right) = \prod_{r=10}^{99} \log_r(r+1)$$

What is the value of $n$

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**Problem 1.11.** If $(x^2+x+1)+(x^2+2x+3)+(x^2+3x+5)+\cdots+(x^2+20x+39)=4500$ Then, the sum of all possible values of $x$ is

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**Problem 1.12.** The value of the product

$$\prod_{n=1}^{98} \frac{n^2+2n}{(n+1)^2} = \frac{a}{b}$$

where $a, b$ are two co-prime integers Then, $a+b=$

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**Problem 1.13.**

$$4^{\frac{x}{y}+\frac{y}{x}} = 32$$

$$\log_3(x-y)+\log_3(x+y)=1$$

Find the value of $x$

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**Problem 1.14.** Evaluate the sum of the expression

$$\sum_{i=1}^{n} i(2)^i$$

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**Problem 1.15.** If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a=b$, then $\frac{a^2}{b^2}=$

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**Problem 1.16.** In a right angle triangle $ABC$, $\sin^2 a + \sin^2 b + \sin^2 c =$

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**Problem 1.17.** If $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$ then $bc + \frac{1}{ck} + \frac{ak}{1+bk} =$

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**Problem 1.18.** if $a\sin\theta + b\cos\theta = b\sin\phi + a\cos\phi = 1$ and $a\tan\theta = b\tan\phi$ then $a+b$ $=$

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**Problem 1.19.** Evaluate the value of the expression

$$\prod_{a=1}^{89} \tan(a^\circ)$$

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**Problem 1.20.** If $\sin(x)+\sin^2 x+\sin^3 x = 1$,then the value of

$$\cos^6 x - 4\cos^4 x + 8\cos^2 x =$$

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*answer.* $x+y$=180 where $x$ is internal angle and $y$ is external angle
$x=4y$ by substitution
$x=144$ and $y=36$
$\frac{180(n-2)}{n}=144$,which means, $n=10$ where $n$ is the number of sides

$\square$

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*answer.* The summition of all external angles must=360,obtuse angle$> 90$
let $n = 3$, $x_1 > 90, x_2 > 90, x_3 > 90$, where $n$ is the number of sides and $x$ is external angle
$x_1 + x_2 + x_3 > 270$ which is possible
let $n = 4$, $x_1 > 90, x_2 > 90, x_3 > 90, x4 > 90$ $x_1 + x_2 + x_3 + x_4 > 360$ which is impossible.
So, the solution is 3 □

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*answer.* Using sine theorem $\frac{a}{}=\frac{b}{}=\frac{c}{}$=m
$a = m$,$b = m$,$c = m$
$\frac{a+c}{b} = m+m\frac{}{m} = +\frac{}{=2\sin\frac{A+C}{2}}+\cos\frac{A-C}{2}\frac{}{2\sin\frac{B}{2}\cos\frac{B}{2}}$
$= 2\sin\frac{\pi}{2} - \frac{B}{2}+\cos\frac{A-C}{2}\frac{}{2\sin\frac{B}{2}\cos\frac{B}{2}=2\cos\frac{B}{2}}+\cos\frac{A-C}{2}\frac{}{2\sin\frac{B}{2}\cos\frac{B}{2}}$
$\frac{\cos\frac{A-C}{2}}{\sin\frac{B}{2}} = \cot\frac{B}{2}$
note that $\cos\frac{A-C}{2} = \cos\frac{B}{2}$
$A = B + c$ which means $A = 90$, or $A + B = C$ which means $C = 90$ □

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*answer.* $+ = 2$
$+ \quad = 2$
$= \frac{b^2+c^2-a^2}{2bc}, = \frac{a^2+c^2-b^2}{2ac}, = \frac{b^2+a^2-c^2}{2ba}$
$a = m$,$b = m$,$c = m$
By substitution you will get the following $2c^2 = 2(b^2 + a^2 - c^2)$
$2c^2 = b^2 + a^2$ □

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*answer.* $\frac{a}{}=\frac{b}{}=\frac{c}{} = m$
a=$m$,$b = m$
$\frac{a-b}{a+b} = \frac{m-m}{m+m} = \frac{-}{+}$
$= \frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \tan\frac{A-B}{2} * \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}} = \tan\frac{A-B}{2}\tan\frac{C}{2}$ □

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Checking if the system is defined for two variables is a hard task, so we shall find the eventual solutions to the system and check directly if the system is defined for them. We shall only write $\left| \begin{array}{l} x+y>0 \\ x-y>0 \end{array} \right.$ for now.

$$\left| \begin{array}{l} \frac{x}{y}+\frac{y}{x}=log_4 32 \\ log_3(x^2-y^2)=1 \\ \frac{x}{y}+\frac{y}{x}=\frac{1}{2}log_2 32 \\ x^2-y^2=3 \\ \frac{x^2+y^2}{xy}=\frac{5}{2} \\ x^2-y^2=3 \\ x^2+y^2=\frac{5}{2}xy \\ x^2-y^2=3 \\ x^2-2xy+y^2=\frac{1}{2}xy \\ (x-y)(x+y)=3 \\ (x-y)^2=\frac{xy}{2} \\ (x-y)(x+y)=3 \end{array} \right.$$
, we divide the first with the square of the second:

$\frac{1}{(x+y)^2}=\frac{xy}{18}$, or $(x+y)^2=\frac{18}{xy}$. We get the system

$$\left| \begin{array}{l} x^2+2xy+y^2=\frac{18}{xy} \\ x^2-2xy+y^2=\frac{xy}{2} \end{array} \right.$$
, we now subtract them

$4xy=\frac{18}{xy}-\frac{xy}{2}$
$\frac{9}{2}xy=\frac{18}{xy}$
$(xy)^2=4$
$xy=\pm 2$. But it can\'t be negative, because $2(x-y)^2=xy$, therefore $xy=2$. Substituting back into the system, we get

$$\left| \begin{array}{l} (x-y)^2=x^2-2xy+y^2=x^2+y^2-4=1 \\ x^2-y^2=3 \\ x^2+y^2=5 \\ x^2-y^2=3 \end{array} \right.$$
, we subtract them:

$2x^2=8$, so $x_1=2$ and $x_2=-2$. $y_1=\frac{2}{x_1}=1$ and $y_2=\frac{2}{x_2}=-1$. We now have to check if they are solutions. $x_1+y_1=3>0$, $x_1-y_1=1>0$, so $(2;1)$ is a solution to the system.
$x_2+y_2=-3<0$, so $(-2;-1)$ is not a solution.
The final solution is *x=2*.

*answer.* □

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*answer.* $\sum_{i=1}^{n} i(2)^i = 2 + 2 * 2^2 + 3 * 2^3 \ldots\ldots n * 2^n$
$\sum_{i=1}^{n}(2)^i + \sum_{i=2}^{n}(2)^i + \sum_{i=3}^{n}(2)^i \ldots\ldots + 2^n$
$2^{n+1} - 2 + 2^{n+1} - 2^2 + 2^{n+1} - 2^3 + \ldots\ldots + 2^{n+1} - 2^n +$
$n2^{n+1} - \sum_{i=1}^{n}(2)^i = n2^{n+1} - (2^{n+1} - 2) = (n-1)2^{n+1} + 2$ □

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*answer.* divide this equation $a\sin^2 x + b\cos^2 x = c \,/\, \cos^2 x$
$a\tan^2 x + b = c\sec^2 x ==> a\tan^2 x + b = c(1 + tan^2 x$
$\tan^2 x = \frac{b-c}{c-a}$
Similarly in the other equation dividing by $sin^2 x$
$\tan^2 y = \frac{a-d}{d-b}$
$a = b ==> a^2\tan^2 x = b^2 tan^2 y$
$\frac{a^2}{b^2} = \frac{\frac{a-d}{d-b}}{\frac{b-c}{c-a}} = \frac{(a-d)(c-a)}{(b-c)(d-b)}$ □

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*answer.* let $C = 90\circ\ ^2 +^2 + =^2 + \sin(90 - A)^2 +$
$=^2 +^2 + = 1 + 1 = 2$ □

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*answer.* $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$ then $bc + \frac{1}{ck} + \frac{ak}{1+bk}$
by substitution, $\frac{}{k^2} + \frac{k}{*k} + \frac{\overline{k}}{1+\frac{*k}{k}}$
$\frac{2}{k^2} + ^1 + \frac{}{1+}$
$\frac{}{k^2} + ^1 = \frac{ak}{k^2} + \frac{1}{ak} = \frac{1}{k}(a + \frac{1}{a})$ □

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*answer.* Given that $a\sin\theta + b\cos\theta = b\sin\phi + a\cos\phi = 1$ $a\tan\theta = b\tan\phi$ divide the first equation by $\cos^2$ you will get
$a\tan^2\theta + b = \sec^2\theta ==> a\tan^2\theta + b = 1 + \tan^2\theta$
$\tan^2\theta = \frac{b-1}{1-a} ==> \tan\theta = \sqrt{\frac{b-1}{1-a}}$
Similarly in the second equation, you will get $\tan\phi = \sqrt{\frac{1-a}{b-1}}$
Using the other given $a\tan\theta = b\tan\phi$
$a\sqrt{\frac{b-1}{1-a}} = b\sqrt{\frac{1-a}{b-1}}$
$\frac{a}{b} = \frac{1-a}{b-1} ==> ab - a = b - ab$
$a + b = 2ab$ □

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*answer.* $\tan 89 = \tan(90-1) = \cot 1$ $tan1 * cot1 = 1$
the final result will be $tan45 = 1$ □

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*answer.* $+\sin^3 x = 1 - \sin^2 x = \cos^2 x$
$(1 + \sin^2 x) = \cos^2 x$ squaring both sides
$\sin^2 x(1 + \sin^2 x)^2 = \cos^4 x$
$(1 - \cos^2 x)(2 - \cos^2 x)^2 = \cos^4 x$
$(1 - \cos^2 x)(4 + \cos^4 x - 4\cos^2 x)$
$4 + \cos^4 x - 4\cos^2 x - 4\cos^2 x - \cos^6 x + 4\cos^4 x = \cos^4 x$
$-\cos^6 x + 4\cos^4 x - 8\cos^2 x + 4 = 0$
$\cos^6 x - 4\cos^4 x + 8\cos^2 x = 4$ □

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